AN OVERVIEW OF A CRANK NICOLSON METHOD TO SOLVE PARABOLIC PARTIAL DIFFERENTIAL EQUATION.

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Abstract: In this paper we have discussed the solving Partial Differential Equation using classical Analytical method as well as the Crank Nicholson method to solve partial differential equation. We compared both the methods and found that decreasing the time step will reduce the error by Crank Nicholson method. The results of the method are obtained by the Scilab programming code.

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INTRODUCTION

Partial differential equation has its application in many engineering problems. Due to the advances in the technology there is a constant search for the proper numerical method to solve a particular PDE problem. Crank Nicolson method is one of the numerical methods to solve a partial differential equation.

Consider the following heat equation with c-constant

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad\qquad (1)$$

With the initial condition u(x,0)=f(x) and the boundary condition u(x_{min},0)=0 and u(x_{max},0)=0.We shall consider a numerical solution that it takes on a grid of x,t values placed over some domain of interest. Once we have these values since 'u' is assumed to be smooth almost everywhere. We can interpolate within the grid to get values for arbitrary x,t.

Suppose the domain we are working is a rectangular with x ranging from x_{min} to x_{max} and 't' ranging from 0 to T. Divide [0,T] into n equal spaced intervals at t values indexed by N=0,1,2,…..n and [x_{min},x_{max}] into I intervals at x values indexed by i=0,1,2……….I. The length of these intervals is k in the time direction and h in the x direction. We seek an approximation to the true values of u at the (n+1)X(I+1) grid points.

Let ui,n denote our approximation at the grid point where x=x_{min}+ih and t=nk. A finite approximation of the equation (1) with 'h' in 'x' direction and 'k' in 't' direction.

$$\frac{1}{k}\left(u_{i,j+1} - u_{i,j}\right) = \frac{c^2}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \qquad\qquad (2)$$

Let $r = \frac{k}{h^2}$

Then the equation(2) is given by

$$u_{i,j+1} = u_{i,j} + r(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \qquad\qquad (3)$$

In Crank Nicolson method the difference quotient on the right hand side of equation(3) is replaced by ½ times the sum two such difference quotients at two time rows.

Therefore the equation (3) can be approximated as

$$u_{i,j+1} - u_{i,j} = \frac{r}{2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + \frac{r}{2}\left(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right).$$

The resulting equation is

$$(2 + 2r)u_{i,j+1} - ru_{i+1,j+1} = (2 - 2r)u_{i,j} + r(u_{i+1,j} + u_{i-1,j}) \qquad (4)$$

Illustration with example

Consider the following problem

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 y}{\partial x^2}$, $0 \le x \le 1$, $t \ge 0$.

$$u(x,0) = sin\pi x \ and \ u(0,t) = u(1,t) = 0. \qquad (5)$$

In Crank Nicolsonscheme we divide the interval 0≤x≤ 1 in equation(5) into n equal intervals we have (n-1) mesh point per time row. Then for j=1 and i=1,2……..n-1 from equation(4) we have a system of n-1 linear equations for 9 unknowns values $u_{1,1}, u_{2,1}, u_{3,1}, \ldots\ldots\ldots u_{n-1,1}$ in the first time row in terms of initial values $u_{1,0}, u_{2,0}, u_{3,0}, \ldots\ldots\ldots u_{n-1,0}$ and the boundary values $u_{0,1} = u_{n,1} = 0$.

Similarly for j=1,j=2 and so on that is for each time row we have to solve such a system of n-1 linear equation resulting form equation (4).

The matrix form of equation (4) is

MU=mb

Where MU=
$$\begin{bmatrix} 2+2r & -r & 0 & . & . & . & 0 \\ -r & 2+2r & -r & 0 & . & . & 0 \\ 0 & -r & 2+2r & -r & 0 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & -r \\ 0 & 0 & . & . & . & -r & 2+2r \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ . \\ . \\ . \\ . \\ u_{n-1,j+1} \end{bmatrix}$$

$$mb=\begin{bmatrix} 2-2r & r & 0 & . & . & . & 0 \\ r & 2-2r & r & 0 & . & . & 0 \\ 0 & r & 2-2r & r & 0 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & r \\ 0 & 0 & . & . & . & r & 2-2r \end{bmatrix}\begin{bmatrix} u_{1,j} \\ u_{2,j} \\ . \\ . \\ . \\ . \\ u_{n-1,j} \end{bmatrix}$$

Convergence of Crank Nicolson method:

This method converges if the following condition is satisfied i.e.,$r=\frac{k}{h^2}\le\frac{1}{2}$.

Solution Using Analytical method

Let$u = XT$

Then by equation (5)

$XT' = X''$ There c=1

By separating the variable

$$\frac{T'}{T}=\frac{X''}{X}=-p^2$$

$$\Rightarrow T'+p^2T=0 \text{ and } X''+p^2X=0.$$
$$\therefore X = a\,cospx + b\,sinpx$$
$$and\ T = ce^{-p^2t}$$

The general solution is

$$u(x,t) = e^{-p^2t}(A\,cospx + B\,sinpx) \qquad (6)$$

The boundary condition u(0,t)=0 is applied to (6)

$$\Rightarrow u(x,t) = A\,sinpx\,e^{-p^2t} \qquad (7)$$

Now the initial condition $u(x,0) = sin\pi x$ is applied to (7)

we get $u(x,t) = e^{-\pi^2t}sin\pi x.$

Solution by Crank Nicolson

$u_{0,j} = 0$. Let us divide the given interval (0,1) into ten equal so that h=0.1 and let k=0.001.

$r = \frac{k}{h^2} = 0.1 \le \frac{1}{2}$ and $u_{i,0} = sin\pi ih$ (i=1,2,3……9)

Now we have a set of nine equations in nine unknowns to solve. The results of the Crank Nicolson method are obtained by Scilab Programming.

Scilab Programming

```
clc
clear
h=0.1;
k=0.001;d=k/(h*h);n1=10;
//INITIALISATION OF THE MATRIX M
fori=1:9;
    l=%pi*(i)*0.1;
t(i)=sin(l)
end
for u=1:6;
m(1,1)=2+2*d;
m(1,2)=-d;
fori=2:n1-2;
m(i,i-1)=-d;
m(i,i)=2+2*d;
m(i,i+1)=-d;
end
m(n1-1,n1-2)=-d;
m(n1-1,n1-1)=2+2*d;
m(1,n1)=(2-2*d)*t(1)+d*t(2);
fori=2:n1-2;
    m(i,n1)=d*t(i-1)+(2-2*d)*t(i)+d*t(i+1);
end
m(n1-1,n1)=d*t(n1-2)+(2-2*d)*t(n1-1);
for p=1:n1-2;
for q=p+1:n1-1;
        k=m(q,p)/m(p,p);
        m(q,:)=m(q,:)-(m(p,:)*k);
end
end
//FINDING U(i,j)
t(n1-1)=m(n1-1,n1)/m(n1-1,n1-1);
for j=n1-2:-1:1;
        s=0;
for k=j+1:n1-1;
        s=s+m(j,k)*t(k);
end
t(j)=(m(j,n1)-s)/m(j,j);
end
w=u*0.001;
printf('t=%f      ',w)
fori=1:n1-1
printf('%f        ',t(i))
end
printf('\n')
end
```

Comparison of Analytical and the Crank Nicolson method

At x=0.5	Analytical	Crank Nicolson	Error
At t=0.006	**0.94250**	**0.942959(k=0.01)**	-4.59×10^{-4}
		0.942958(k=0.02)	-4.58×10^{-4}
At t=0.008	**0.924079**	**0.924678(k=0.001)**	-5.99×10^{-4}
		0.924676(k=0.02)	-5.97×10^{-4}

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